Optimization Problems - Classwork

Many times in life we are asked to do an optimization problem - that is, find the largest or smallest value of some quantity that will fufill a need. Typical situations are:

- find the route which will minimize the time it takes me to get to school.
- build a structure using the least amount of material.
- build a structure costing the least amount of money.
- build a yard enclosing the most amount of space.
- find the least medication one should take to help a medical problem.
- find how the most one should charge for a CD in order to make as much money as possible.

All of these situations have something in common - they are all trying to maximize or minimize some quantity. This lends itself to a calculus solution. We have spent the better part of last month trying to find maximum and minimum values of functions. In every optimization problem, you are always looking for a quantity to be maximimed or minimized. So in solving word problems, you must look carefully for certain words among all the verbiage. Look for words like "minimize area", "smallest volume", "least amount of time", "shortest distance", "cheapest price." On the following pages, there are a wealth of problems. Quickly examine each and underline the key words which tell you what kind of problem it is.

Methods for Solving Optimization Problems

1. Assign variables to all given quantities and quantities to be determined. Don't be afraid to use letters you usually do not use (p,m,g, etc.). When feasible, make a sketch of the problem.

2. Making a chart of possible answers allows you to see a relationship between variables. While not necessary, it is helpful.

3. Write a "primary" equation for the quantity you found that needs to be maximized or minimized

   Area of Rectangle = length • width
   Distance = rate • time
   Volume of rectangular solid = length • width • height
   Hypotenuse = $\sqrt{x^2 + y^2}$
   Perimeter of rectangle = 2• length + 2• width
   Volume of cylinder = π(radius^2) • height

4. Reduce the right side of this "primary equation" to one having a single variable. If there is more than one variable on the right side, you must write a "secondary" equation (a restriction or constraint) relating the variables of the primary equation.

5. Take the derivative of the equation and **set equal to zero.** If you get more than one answer, make a sign chart to determine whether it represents a maximum or minimum. Pay attention to whether that value makes sense. Time is rarely negative (it can't take negative 7 hours to run a race). You cannot use more than you have (you can't have a length of 8 feet when you only have 6 feet of fencing).

6. **Be sure that you answer the question that is asked.** If you are asked to find a minimum or maximum value of some quantity, you must plug your answer from (4) into your primary equation.

7. If you are to find a maximum or minimum on a closed interval, you must test the endpoints as well. Make sure your work is clear.

8. You can verify your answers by graphing your primary equation with one variable on the calculator. Use your 2nd CALC maximum or minimum function.

Example 1) Two numbers add up to 40. Find the numbers that maximize their product.

Smaller Number						
Larger Number						
Product						

Primary *Secondary*

Example 2) A rectangle has a perimeter of 71 feet. What length and width should it have so that its area is a maximum? What is this maximum area?

Width						
Length						
Area						

Primary *Secondary*

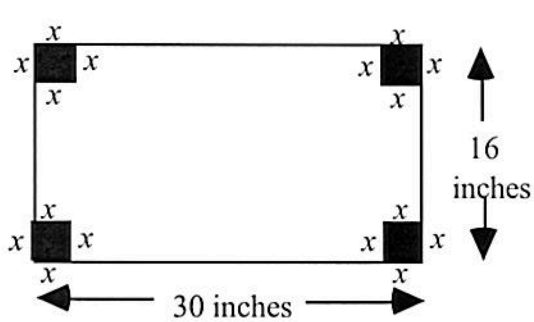
Example 3) Find two positive numbers that minimize the sum of twice the first number plus the second if the product of the two numbers is 288.

First Number						
Second Number						
Sum						

Primary *Secondary*

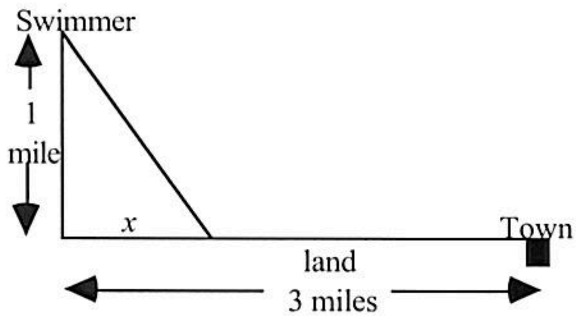
Example 4) An open box is to be made from a piece of metal 16 by 30 inches by cutting out squares of equal size from the corners and bending up the sides. What size square should be cut out to create a box with greatest volume. What is the maximum volume as well?

Primary



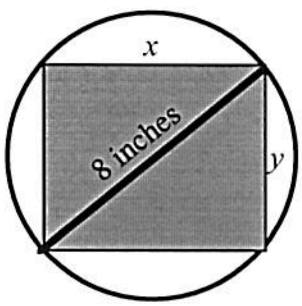
Example 5) I am 1 mile in the ocean and wish to get to a town 3 miles down the coast which is very rocky. I need to swim to the shore and then walk along the shore. What point should I swim to along the shoreline so that the time it takes to get to town is a minimum? I swim at 2 mph and walk at 4 mph.

Primary



Example 6) . Find the dimensions of the largest area rectangle which can be inscribed into a circle of radius 4 inches.

primary *secondary*



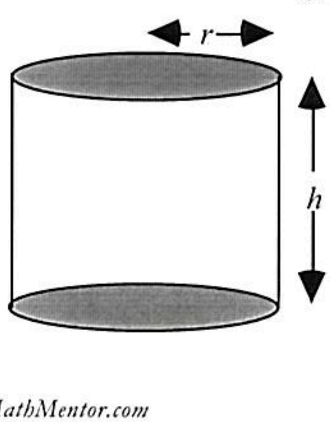
How would this problem change if the radius were r inches?

Example 7) A 6-oz. can of Friskies Cat food contains a volume of approximately 14.5 cubic inches. How should the can be constructed so that the material made to make the can is a minimum?

Primary *Secondary*

Surface Area = Area of side + Area of Top & bottom $V = \pi r^2 h$

$S = 2\pi rh + 2\pi r^2$



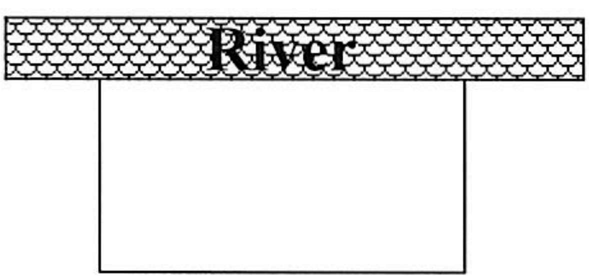
Optimization Problems - Homework

1. Find two numbers whose sum is 10 for which the sum of their squares is a minimum.

2. Find nonnegative numbers x and y whose sum is 75 and for which the value of xy^2 is as large as possible.

3. A ball is thrown straight up in the air from ground level. Its height after t seconds is given by $s(t) = -16t^2 + 50t$. When does the ball reach it maximum height? What is its maximum height?

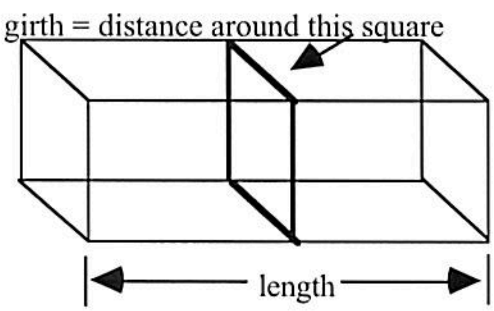
4. A farmer has 2,000 feet of fencing to enclose a pasture area. The field will be in the shape of a rectangle and will be placed against a river where there is no fencing needed. What is the largest area field that can be created and what are its dimensions?



5. A fisheries biologist is stocking fish in a lake. She knows that when there are n fish per unit of water, the average weight of each fish will be $W(n) = 500 - 2n$, measured in grams. What is the value of n that will maximize the total fish weight after one season. *Hint: Total Weight = number of fish • average weight of a fish.*

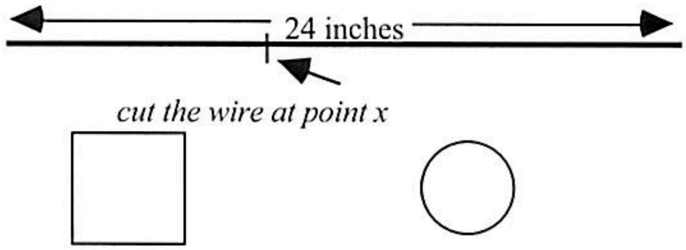
6. The size of a population of bacteria introduced to a food grows according to the formula $P(t) = \dfrac{6000t}{60 + t^2}$ where t is measured in weeks. Determine when the bacteria will reach its maximum size. What is the maximum size of the population?

7. The U.S. Postal Service will accept a box for domestic shipping only if the sum of the length and the girth (distance around) does not exceed 108 inches. Find the dimensions of the largest volume box with a square end that can be sent.



8. Blood pressure in a patient will drop by an amount $D(x)$ where $D(x) = 0.025x^2(30 - x)$ where x is the amount of drug injected in cm^3. Find the dosage that provides the greatest drop in blood pressure. What is the drop in blood pressure?

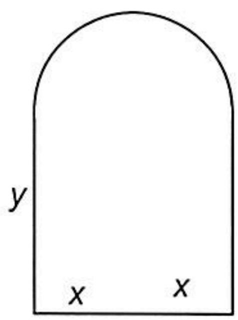
9. A wire 24 inches long is cut into two pieces. One piece is to be shaped into a square and the other piece into a circle. Where should the wire be cut to maximize the total area enclosed by the square and circle?



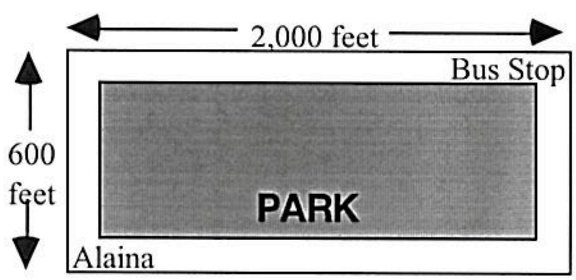
Let x be the point where the cut is made. Assume the square is on the left and the circle on the right. Complete the chart.

x	4	8	12	20	x
Area square					
Area circle					
Total area					

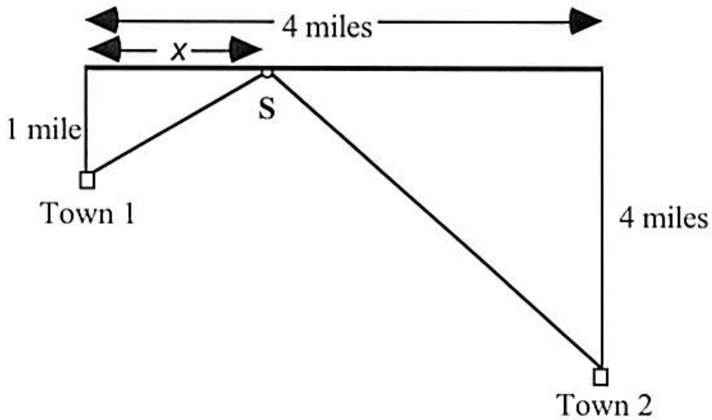
10. A designer of custom windows wishes to build a Norman Window with a total outside perimeter of 60 feet. How should the window be designed to maximize the area of the window. A Norman Window contains a rectangle bordered above by a semicircle.



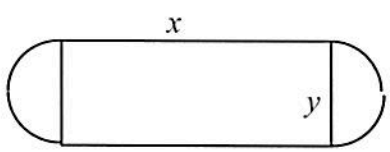
11. Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2,000 feet east and 600 feet north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 feet/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec (dogs are walked here, so she must move with care). What path will get her to the bus stop the fastest.



12. On the same side of a straight river are two towns, and the townspeople want to build a pumping station, **S**, that supplies water to them. The pumping station is to be at the river's edge with pipes extending straight to the two towns. The distances are shown in the figure below. Where should the pumping station be located to minimize the total length of pipe?

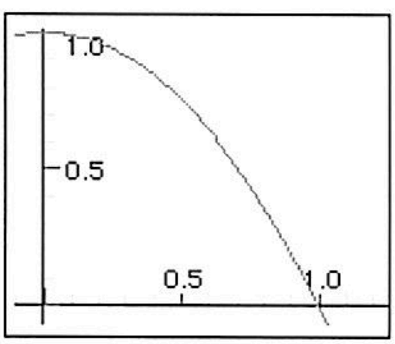


13. A physical fitness room consists of a rectangular region with a semicircle on each end. If the perimeter of the room is to be a 200-meter running track, find the dimensions that will make the area of the rectangular region as large as possible.



Total distance around track = 200 feet

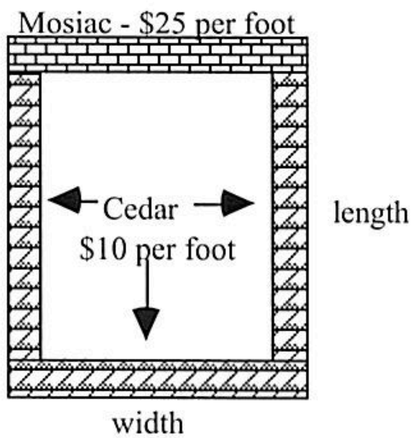
14. Below is the graph of $y=1-x^2$. Find the point on this curve which is closest to the origin. (*Remember, you need a primary equation. What is it that you wish to minimize?*)



Economic Optimization Problems - Classwork

Example 1) A trucking company has determined that the cost per hour to operate a single truck is given by $C(s) = 0.0001s^2 - 0.01s + 12$ where s is the speed that the truck travels. At what speed is the total cost per hour a minimum? What is the hourly cost to operate the truck?

Example 2) A nursery wants to add a 1,000-square-foot rectangular area to its greenhouse to sell seedlings. For aesthetic reasons, they have decided to border the area on three sides by cedar siding at a cost of \$10 per foot. The remaining side is to be a wall with a brick mosaic that costs \$25 per foot. What should the dimensions of the sides be so that the cost of the project will be minimized?

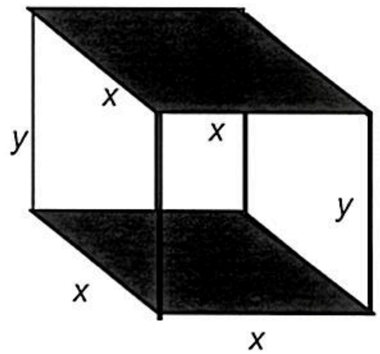


Width	Length	Cedar Cost	Mosaic Cost	Total Cost
x				

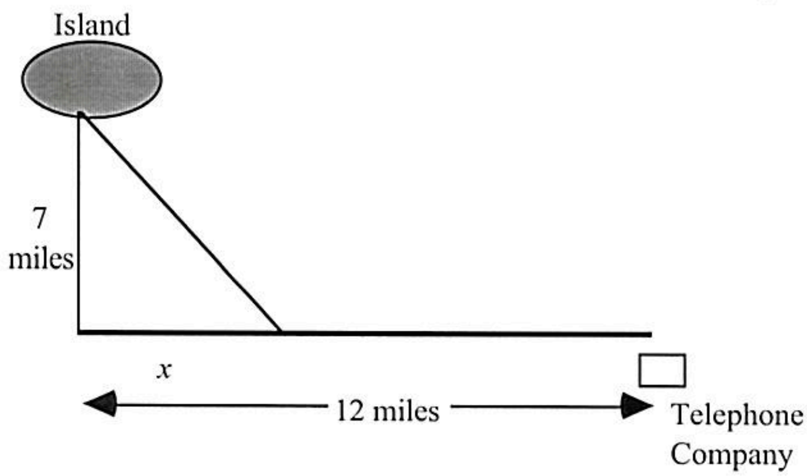
Example 3) A real estate company owns 100 apartments in New York City. At \$1,000 per month, each apartment can be rented. However, for each \$50 increase, there will be two additional vacancies. How much should the real estate company charge for rent to maximize its revenues?

$50 increases	Rent	Apts. rented	Revenue
0			
1			
2			
3			
4			
50			
x			

Example 4) A closed box with a square base is to have a volume of 1,800 in^3. The material for the top and bottom of the box costs \$3 per square inch while the material for the sides cost \$1 per square inch. Find the dimensions of the box that will lead to the minimum total cost. What is the minimum total cost?



Example 5. A telephone wire is to be laid from the telephone company to an island 7 miles off shore at a cost of \$200,000 per mile along the shoreline and \$300,000 per mile under the sea. How should the wire be laid at the least expensive cost if the distance along the shoreline is 12 miles. What is that cost?



x	water cost	land cost	total cost
0			
12			
6			

Example 6) A small television company estimates that the cost (in dollars) of producing x units of a certain product is given by $C = 800 + .04x + .00002x^2$. Find the production level that minimizes the *average cost per unit.*

Units	Cost	Average Cost
100		
1,000		
5,000		
10,000		

Economic Optimization Problems - Homework

1. The profit for Ace Advertising Co. is $P = 230 + 20s - \frac{1}{2}s^2$ where s is the amount (in hundreds of dollars) spent on advertising. What amount of advertising gives the maximum profit?

2. North American Van Lines calculates charges for delivery according to the following rules.

   Fuel cost $= \dfrac{v^2}{120}$ per hour   Driver cost = \$30 per hour

   Find the speed v that a truck should travel in order to minimze costs for a trip of 120 miles. *Hint: remember that rate • time = distance. Make a chart of possible speeds v and total costs.*

3. Normally a pear tree will produce 30 bushels of pears per tree when 20 (or fewer) pear trees are planted per acre. However, for each additional pear tree planted above 20 trees per acre, the yield per tree will fall by one bushel per tree (why?). How many trees ahould be planted per acre to maximize the total yield? *Hint: Make a chart like the Apartment Housing sample problem.*

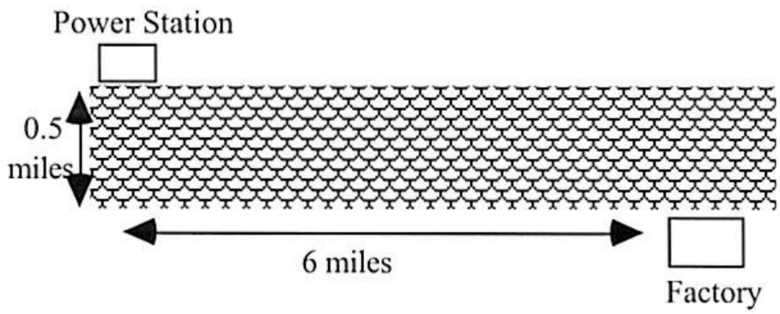
4. Midas Muffler charges \$28 to replace a muffler. At this rate, the company replaces 75,000 mufflers per week nationally. For each additional dollar that the company charges, it tends to lose 1,000 customers a week. For each dollar the company subtracts from the \$28, the company gains 1,000 per week. How much should Midas charge to change a muffler in order to maximize their revenue? What would that revenue be? *Hint: Make a chart like the Apartment Housing sample problem.*

5. A concert promoter knows that 5,000 people will attend an event with tickets set at \$10. For each dollar less in ticket price, an additional 1,000 tickets will be sold. What should the price of a ticket be in order to maximize the total receipts. *Hint: Make a chart like the Apartment Housing sample problem.*

6. A travel agent is offering charter holidays in the Bahamas for college students. For groups of size up to 100, the fare is \$1,000 per student. For larger groups, the fare per person decreases by \$5 for each additional person in excess of 100. Find the size of the group that will maximize the travel agent's revenues. *Hint: Make a chart like the Apartment Housing sample problem.*

7. A real estate office handles 50 apartment units. When the rent is \$540 per month, all units are occupied. However, on the average, for each \$30 increase in rent, one unit becomes vacant. Each occupied unit requires an average of \$36 per month for service and repairs. What rent should be charged to realize the most profit?

8. A power station is on one side of a river that is .5 mile wide, and a factory is 6 miles downstream on the other side. It costs \$6,000 per mile to run power lines overland and \$8,000 per mile to run them underwater. Find the most economical path to lay transmission lines from the station to the factory.



9. A rectangular area is to be fenced in using two types of fencing. The front and back uses fencing costing \$5 a foot while the sides uses fencing costing \$4 a foot. If the area of the rectangle must contain 500 square feet, what should be the dimensions of the rectangle in order to keep the cost a minimum?

10. The same rectangular area is to be built, but now the builder has only \$800 to spend. What is the *largest area* that can be fenced in using the same two types of fencing mentioned above.

OPTIMIZATION – PART 2 NAME________________________________

1. The operating cost of a truck is $12+\frac{x}{6}$ cents per mile when the truck travels x miles per hour. If the driver earns \$6 per hour, what is the most economical speed to operate the truck on a 400 mile turnpike? Due to construction, the truck can only travel between 35 and 60 miles per hour.

2. A furniture business rents chairs for conferences. A contract is drawn to rent and deliver up to 400 chairs for a particular meeting. The exact number would be determined by the customer later. The price will be \$90 per chair up to 300 chairs. If the order goes above 300 chairs, the price would be reduced by \$0.25 per chair for every additional chair ordered above 300. This reduced price would be applied to the entire order. Determine the largest and smallest revenues this business can make under this contract.

3. The speed of traffic through the Lincoln Tunnel depends on the density of the traffic. Let S be the speed in miles per hour and D be the density in vehicles per mile. The relationship between S and D is approximately $S = 42 - \frac{D}{3}$ for $D \leq 100$. Find the density that will maximize the hourly flow.

4. A commercial cattle company currently allows 20 steer per acre of grazing land. On average a steer weighs 2000 pounds at the market. Estimates by the Department of Agriculture indicate that the average weight per steer will be reduced by 50 pounds for each additional steer added per acre of grazing land. How many steer per acre should be allowed in order to optimize the total market weight of the cattle?

Name:______________________________ Date:__________

' Calc Intro to Max-Min Problems

1. My dachshund, Rocky, needs a place to play. I purchased 100 feet of fencing to use to make him an enclosed rectangular play area. What should the dimensions of this play area be in order for him to have the largest play area possible? What is the resulting maximum area?

2. Now that Rocky has a place to play, he needs to have a place to keep his toys. I want to make an open-top box so that he can reach in and get the toys out himself. I want to use a 20-by-25 inch sheet of tin to make this box. I will cut congruent squares of side length x from the corners of the sheet of tin and bend up the sides. How large should the squares be to make the box hold as much as possible? What is the resulting volume?

Name: ____________________ Date: __________

Calculus Max-Min Problems (Sheet 2)

1. A rectangle has its base on the x-axis and its two upper corners on the parabola $y = 12 - x^2$. What is the largest possible area of the rectangle?

2. An open rectangular box is to be made from a 9×12 inch piece of tin by cutting squares of side x inches from the corners and folding up the sides. What should x be to maximize the volume of the box?

3. A 384-square-meter plot of land is to be enclosed by a fence and divided into two equal parts by another fence parallel to one pair of sides. What dimensions of the outer rectangle will minimize the amount of fence used?

4. What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material?

Name: ______________________ Date: ________

Calculus | More Max-Min Problems (Sheet 3)

1. A farmer has 600 m of fencing with which he plans to enclose a rectangular pen adjacent to a long existing wall. He will use the wall for one side of the pen and the available fencing for the three remaining sides. What is the maximum area that he can enclose this way?

2. The sum of two positive numbers is 48. What is the smallest possible value of the sum of their squares?

3. A rectangular box has a square base whose edge is at least 1 cm, and its total surface area is 600 cm^2. What is the largest possible volume that such a box can have?

4. Three large squares of tin, each of edge length 1 m, have four equal small squares cut from their corners. All twelve resulting small squares are to be the same size. The three large cross-shaped pieces are then folded and welded to make boxes with no tops, and the twelve small squares are used to make two cubes. How should this be done to maximize the total volume of all five boxes?

5. You have been asked to design a one-liter oil can shaped like a right circular cylinder. What dimensions should you use for the can to make the surface area as small as possible? (1 liter $= 1000\ \text{cm}^3$)

Name:____________________________________ Date:____________

Calc Min-Max Problems (Sheet 4)

1971- AB 4

Find the area of the largest rectangle (with sides parallel to the coordinate axes) that can be inscribed in the region enclosed by the graphs of $f(x) = 18 - x^2$ and $g(x) = 2x^2 - 9$.

1972- AB 4

A man has 340 yards of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide and the other a square. The square field must contain at least 100 square yards and the rectangular field must contain at least 800 square yards.

(a) If *x* is the width of the rectangular field, what are the maximum and minimum possible values of *x*?

(b) What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.

1973- AB 6

A manufacturer finds it costs him $x^2 + 5x + 7$ dollars to produce *x* tons of an item. At production levels above 3 tons, he must hire additional workers, and his costs increase by $3(x - 3)$ dollars on his total production. If the price he receives is \$13 per ton regardless of how much he manufactures and if he has a plant capacity of 10 tons, what level of output maximizes his profit?

1982- AB 6

A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs \$10 per square meter for the base and \$5 per square meter for the sides, what is the cost of the least expensive tank?